COLLEGE STUDENTS' MEANINGS FOR BRACKETS AND PARENTHESES IN ALGEBRA

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Literature Review

- Limited research on students' understandings of parentheses:
- Focused on what students **do** with parentheses, not their conceptualizations of what parentheses **mean**.
- Most research with K-12 students, not college students.

Methods

- Open-ended questions and interviews.
  - 124 open-ended responses students at an urban community college (18 different courses: developmental elementary algebra to linear algebra).

- Analyzed using thematic analysis (Braun & Clarke, 2006).
  - Influenced by theoretical stance attuned to noticing similarities in patterns of students’ responses with theory on extracted and stipulated definitions (Edwards & Ward, 2004) and computational and structural (Sfard, 1995) conceptions.
  - Led to more nuanced emergent coding scheme of responses for definitions of parentheses.

Types of Conceptions of Parentheses

Extracted computational view: A cue to a procedure. Procedure is not consistent in all contexts with stipulated order of operations and other symbolic conventions (e.g., multiply regardless of actual operations).

Stipulated normative computational view: A cue to follow stipulated order of operations (i.e., simplify what is inside the innermost brackets first).

Structural view: Demarcating a unified sub-expression within a larger expression or equation; can be thought of as an object in and of itself. This may be the reification of the process of the stipulated order of operations.

Different non-normative computational conceptions of parentheses

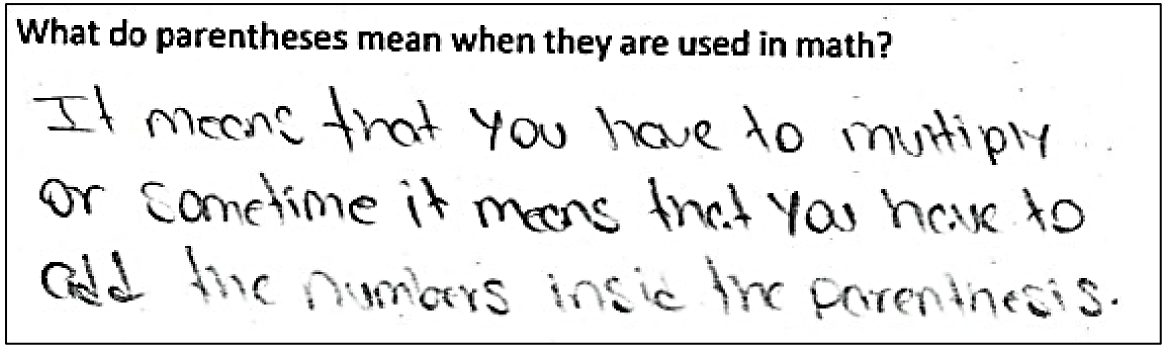
It was common for parentheses to cue various non-normative procedures, though they tended to fall mostly into one of three categories:

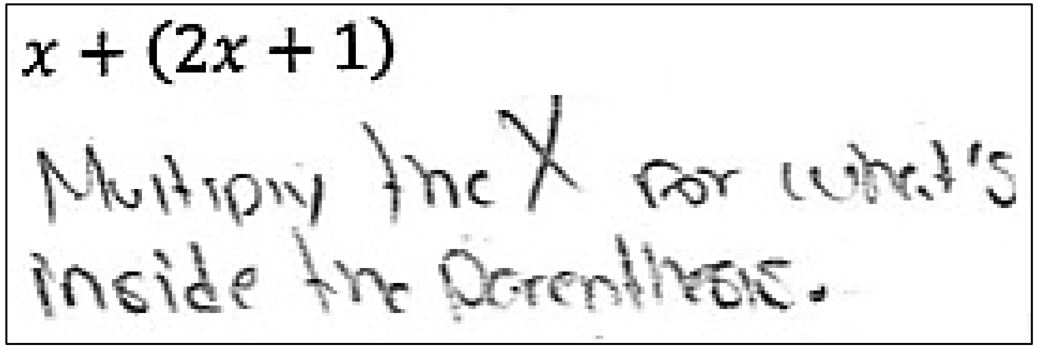
- “Multiplication”
- “do first”
- “ease of reading”

We describe these one at a time…

“Multiplication” Conception of Parentheses

- The most common response among students.
- Students rarely recognized distinction between parentheses and multiplication. Parentheses often co-occur with concatenation (which is what actually indicates multiplication).
  - May be an extracted definition taken from instructional experiences (tasks where parentheses and multiplication co-occur).
  - Instructors may also sometimes say “parentheses mean to multiply”).
- Not mathematically normative; but a rational reaction to instructional practices.
- May lead to: 1) incorrect results if operation next to parentheses is not multiplication; 2) fundamental misunderstanding of role of parentheses.





"Do First" Conception of Parentheses

Refers to precedence of parentheses in order of operations. Students may have normative or non-normative "do first" conceptions—we illustrate a non-normative one here, where the student computes in an incorrect order that appears to be cued by presence of parentheses.

(Interview) Gamma attempting to simplify $3 - 2\cdot(8 - 3^2)$:

So, I try in my head, so they break it down, PEMDAS, so parentheses first, so I did like eight minus three is five and I know that's going to leave me with five squared. So, I just left that as it is and looked over here and seen what has five to the second exponent. And realized, you know, okay that's the same because that would be my next step is to solve the parentheses…. Because PEMDAS I solved what's in the parentheses first and then looked at the exponents and then that's pretty much how I saw it.

- The student correctly stated the order of operations, so knowledge of this does not seem to be the issue; parentheses cue work that contradicts their stated order of operations.
- Parentheses cues work inside them to be done first. Because exponents come after parentheses, they do 8-3 first.
- Subtraction inside parentheses computed to "get rid of" parentheses before doing exponent inside the parentheses itself (perhaps because parentheses are viewed as necessitated by subtraction but not the exponent?).
- Parentheses appear to be interpreted as a cue to a particular (incorrect) computational action:

“Do First” Conception of Parentheses

(Interview) Student was asked what was being multiplied by the 2 in $2\cdot(3x - 5)$:
Gamma attempted to simplify $2\cdot(3x - 5)$ in order to answer this question:

When I think of parentheses it's something that has to be done first. In this particular problem I feel like you have to distribute because you have x there. So, it's like you can't solve 3x minus five. I don't think you could just like get an answer from that. You have to solve what's inside of the parentheses. So, what's in the parentheses is 3x minus five. So, in order for me to solve that I must distribute the two that's outside into that equation. I just think of PEMDAS, you have got to do parentheses first, and then exponents, and down the line. So, I just look at a question, I know I have to do something with the parentheses first.

- Able to use “do first” conception of parentheses to simplify correctly, but still used computational approach (focused on “solving”, even though that is not what has been asked).
- Many other students have “distributed” inappropriately in other problems to “get rid of” parentheses (e.g., when there is an exponent outside the parentheses instead of a coefficient)—might Gamma do something similar? Unclear. But may be related.

“Ease of Reading” Conception of Parentheses

“Ease of reading” Conception:

“Parentheses are unnecessary symbols which can always be removed.”

Common Student Justification: parentheses make expressions/equations “easier to read”.

- Related to “bracket ignoring” procedure (Gunnarsson, et al., 2016) with middle school students, but we are interested in shedding more light on **why** they might ignore the bracket.

“Ease of Reading” Conception of Parentheses

What do parentheses mean when they are used in math?

They are used to denote modificatis to normal numbers or equation.

Give at least one example that uses parentheses and explain what the parentheses mean in that example:

$4-(2-2)+4$

$4-0+4$

$=8$

This show that when the bracket is removed it makes it easier for the equation to be solved

Can parentheses be used in math to mean anything else?

Circle one: Yes No I don’t know

Please explain how you know.

It only allows the equation make sense and nothing else

Tau: parentheses make it “easier for the equation to be solved” or “allows the equation to make sense”.

- In arithmetic example, calculations appear to be correct. They “remove” the parentheses by simplifying what is inside them first.

“Ease of Reading” Conception of Parentheses

Simplify completely: $(3x-1)-(3x^2+5x-7)$ **Step 1:** Remove the bracket $3x-1-3x^2+5x-7)$	This helps for easier solution
Step 2: put like terms together $3x+3x^2+5x+1-7$	
Step 3: $6x^2+-1$	

Simplify completely: $(3x-1)\cdot(3x^2+5x-7)$ **Step 1:** Remove bracket $3x-1\times3x^2+5x-7$	It is the meaning of parenthesis.

But, in algebra tasks, Tau literally just removes parentheses when simplifying, regardless of details of the problem.

- Justification given: “Remove the bracket. This helps with easier solution.” And “Remove bracket…It is the meaning of parentheses.”
- Likely extracted from prior mathematical experiences not just that it is desirable to find valid mathematical ways to “get rid of the bracket” but that this process is actually simply to remove the bracket and that parentheses are superfluous.

Combinations of Extracted and Stipulated Computational Views

(Interview) Epsilon was asked what was being subtracted in $3x^2 - 2(x^2+1)$:

Interviewer:	*What do the parentheses mean here in this expression?*
Epsilon:	*Multiplication.*
Interviewer:	*Do you know what the order of operations is?*
Epsilon:	*PEMDAS.*
Interviewer:	*How does the order of operations help you to understand what is being subtracted in this expression?*
Epsilon:	*So, first you have to deal with the parentheses. So, then that means that everything with the parentheses is like together in a box.*

- Epsilon thinks of multiplication first when asked what parentheses “mean”.
- Also exhibits “do first” notion of parentheses when asked how order of operations relates.
  - We can’t determine for certain whether or not it is normative.
- Do-first conceptions is Immediately followed with structural grouping interpretation of parentheses.
  - May be reifying the order-of-operations into the grouping object.

Combinations of Extracted and Stipulated Computational Views

(Interview) Epsilon on meanings of parentheses. Asked whether $\sqrt{(3xy)^2}$ and $\left(\sqrt{3xy}\right)^2$ represent the same operations:

[Parentheses] don't really mean like multiplication per say in this instance. So, in the first one the parentheses is the expression of 3xy to the second power. So, it's distributing the- Well, in both instance it's distributing the second power, but it's just where it's placed. I'm sorry, I don't know if I'm making any sense, but because of where the parentheses are placed it just changes the meaning.

So, here [pointing to first expression] you have to deal with the parentheses first. So, in the first equation you have to distribute the second power to the 3xy and then take the square root. And then in the second equation you have to take the square root first of 3xy and then square it and those would result in different answers probably.

- For Epsilon, parentheses appear to be a cue to distribute the exponent, or even literally "mean" distributing.
  - Here this is done normatively; unclear if this would be the case in expressions with other structures (e.g., distributing exponents over addition?)
- Work is correct but entirely computational rather than structural (might a structural approach make the explanation easier?)

Grouping View of Parentheses

Conceptualizing parentheses structurally as a grouping mechanism.

(Interview) Theta asked the result of substituting 2y in for x into $2x^2 – 7x+3$:

> *If you're putting something in for something else, I always usually keep parentheses around it to make sure that I maintain whatever structure the original function had... especially in this one [the first term] because x is being multiplied, and squared, the parentheses really make a difference because if we don't have them, we could get a different, the wrong answer…. you don't need [the parentheses in -7(2y)] as long as you make sure you multiply the -7 by 2.*

- Theta mentions structure explicitly, and explains how removing the parentheses gives the expression a different syntactic meaning, because of the order of operations.
- They appear to have linked their structural view of the parentheses to their conception of the order of operations, suggesting that it may be a reification of this process.

Implications & Conclusions

- Responses suggest that students extract non-normative meanings of parentheses in algebraic syntax.
  - This may impact students' computational work.
- Unintended consequences of language or examples used by instructors/textbooks: parentheses "mean multiplication"; various computations cued to "get rid of" parentheses in order to "do them first"; or that parentheses are always superfluous.
  - Precise meaning of "Do the parentheses first" may need to be articulated more clearly to students.
  - Provide a variety of examples where parentheses are and are not extraneous to support students.
- Future research: explore how students reify the normative "do first" view of parentheses into a grouping conception of parentheses.
- Future research: explore prevalence of various views among different student groups and relationship to computational work.

Thank you! Any Questions?

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