3.7 Rational Functions

A rational function is a quotient of polynomial functions.

$$f(x) = \frac{N(x)}{D(x)}$$

Where $N(x)$ and $D(x)$ are polynomials and $D(x) \neq 0$.

Examples of polynomial functions are:

$$f(x) = \frac{2x+1}{x-3}$$

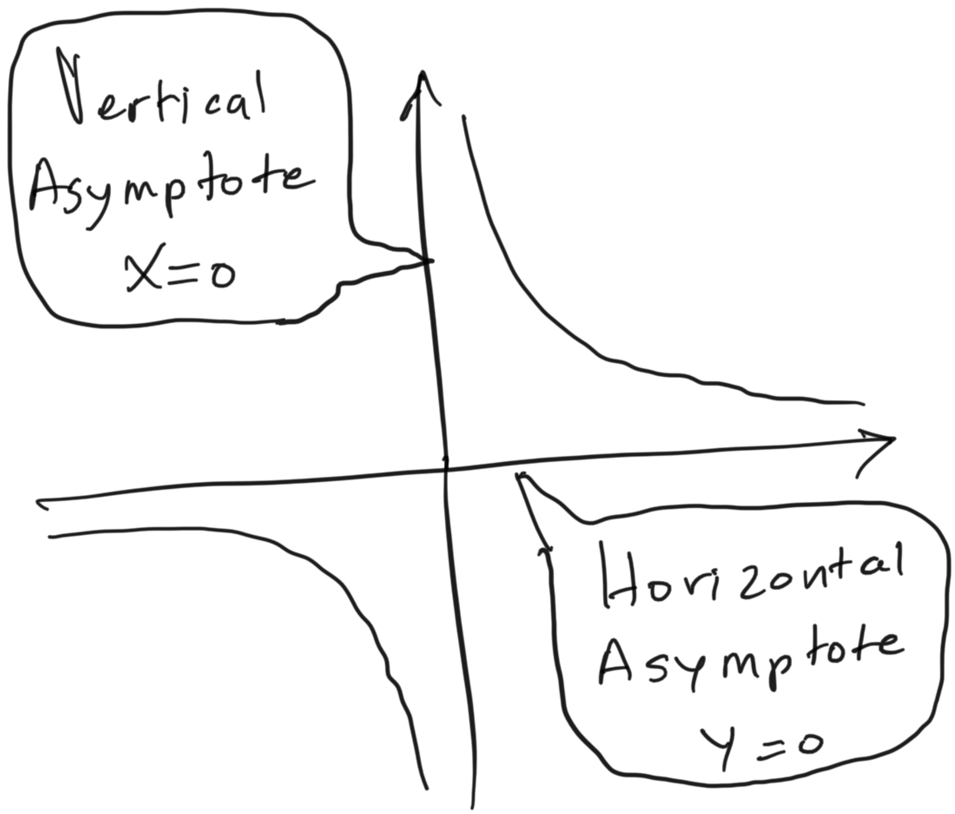
$$g(x) = \frac{x+1}{x^2-5x+6}$$

$$h(x) = \frac{3x}{x^2+1}$$

Example: Given $f(x) = \frac{1}{x}$ state its Domain and sketch its graph.

Solution: $f(x) = \frac{1}{x}$

Domain = All real numbers except 0.



x	−0,5	−0.5	0	0.001	0.5
	2	−1000	DNE	1000	2

$f(x)$

- The line $x=a$ is a vertical Asymptote (V.A.) of the graph of f when $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.

- The line $y=b$ is a Horizontal Asymptote (H.A.) of the graph of f when $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$

Method to find Vertical Asymptotes

Given $f(x) = \dfrac{N(x)}{D(x)}$ in simplified form, set

Simplified

$D(x) = 0$ and solve for x

Examples:

Find the vertical Asymptote(s

1) $f(x) = \dfrac{3}{x-4}$

Solution

set $x - 4 = 0$

$+4 \quad +4$

$x = 4$

then V.A : $x = 4$

2) $g(x) = \dfrac{x-7}{x^2+2x-8}$

Solution:

Set $x^2 + 2x - 8 = 0$

Factor

$(x+4)(x-2)=0$

$$x+4=0 \qquad x-2=0$$
$$-4 \quad -4 \qquad +2 \quad +2$$
$$x=-4 \qquad x=2$$

Then V.A: $x=2$ and $x=-4$

Method to find Horizontal Asymptotes

Given $f(x)=\dfrac{N(x)}{D(x)}=\dfrac{a_n x^n+\dots a_0}{b_m x^m+\dots b_m}$

in simplified form

Compare the Degrees of $N(x)$ and $D(x)$

- If $n<m$ then H.A: $y=0$
- If $n=m$ then H.A: $y=\dfrac{a_n}{b_m}$
- If $n>m$ then No H.A.

Examples: Find the H.A. of the given functions

Degree 1

1) $f(x) = \dfrac{2x^{1}+3}{5x^{2}+2x-4}$

Degree 2

$1 < 2$

Then H.A: $y=0$

Degree 1

2) $g(x) = \dfrac{3x^{1}+11}{7x^{1}-5}$ Degree 1

$1 = 1$

Then H.A.: $y = \dfrac{3}{7}$

Degree 3

3) $h(x) = \dfrac{5x^{3}-2x+3}{2x^{2}-x+2}$ Degree

$3 > 2$

Then NO H.A.

... Graph of a

Sketching the

Rational function

Method:

1. Factor and simplify
2. Find the Y-intercept
   (Let $x=0$ and solve for y)
3. Find the X-intercept
   (Let $y=0$ and solve for x)
4. Find V.A (s)
5. Find H.A (s)
6. Use test points and use smooth curves to complete the graph.

Example; Sketch the graph of

$$f(x) = \frac{x^2}{x^2-x-6}$$

Solution

$$x^2$$

$$f(x) = \frac{\quad}{(x+2)(x-3)}$$

<u>Y-intereept</u>

Let $x=0$ $\qquad y = \frac{0^2}{0^2-0-6} = \frac{0}{-6} =$

then $Y\text{-int} = (0,0)$

<u>X-intercept</u>

Let $y=0$ $\qquad 0 = \frac{x^2}{x^2-x-6}$

$$0 = x^2$$

$$0 = x$$

then $x\text{-int} = (0,0)$

<u>V.A.</u>

$$f(x) = \frac{x^2}{x^2-x-6} = \frac{x^2}{(x+2)(x-3)}$$

Set denominator $= 0$

$$(x+2)(x-3) = 0$$

$$\begin{array}{ll} x+2=0 & x-3=0 \\ -2 \quad -2 & +3 \quad +3 \end{array}$$

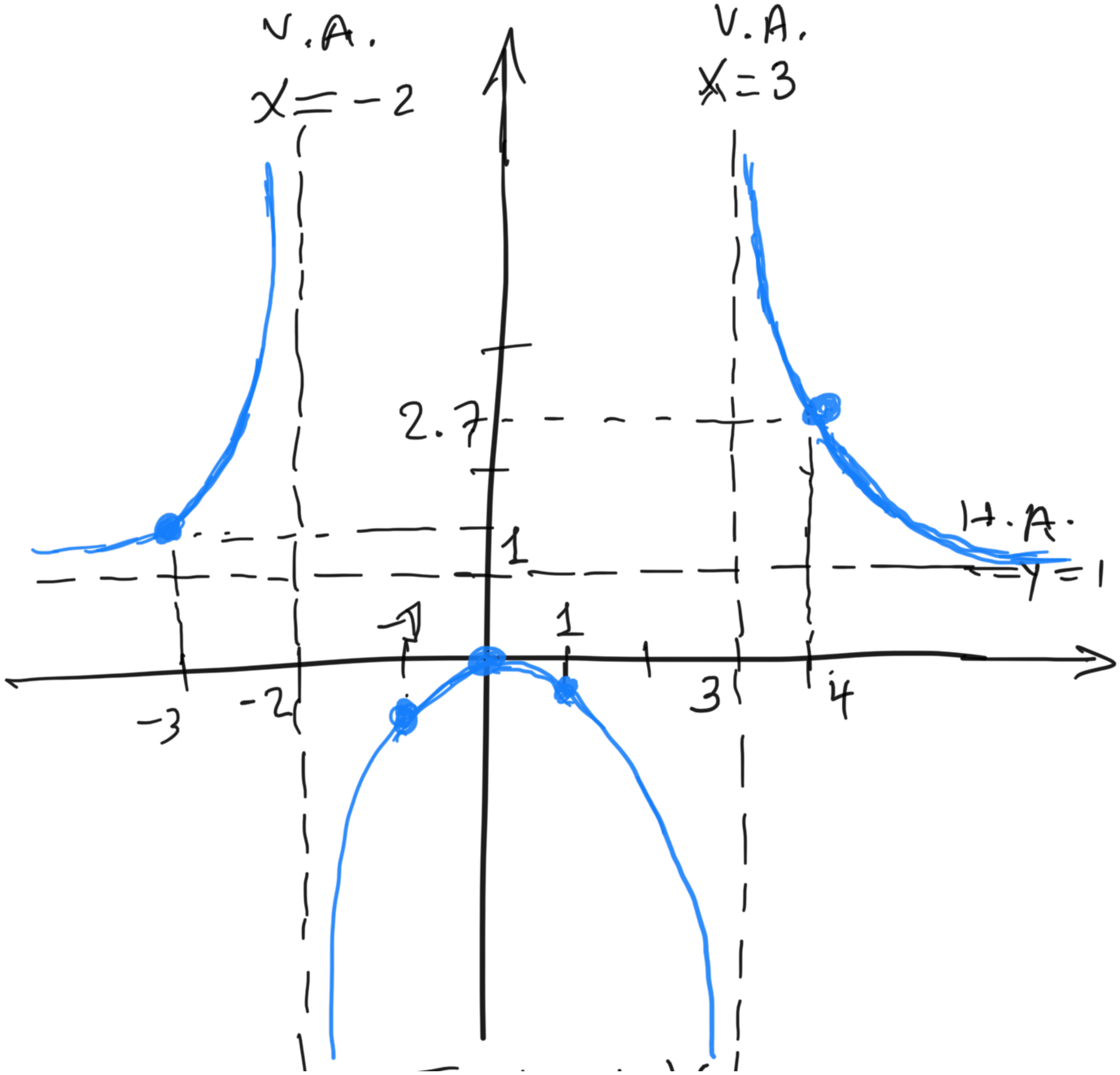
V.A: $x=-2$, $x=3$

H.A.

Since degrees of Numerator and denominator are the same

then $y=\frac{1}{1}=1$

H.A.: $y=1$



Test points

Test point $x = -3$

$$f(-3) = \frac{(-3)^2}{(-3)^2 - (-3) - 6}$$

$$= \frac{9}{6}$$

$$= 1.5$$

$x = -1$, $x = 1$

$f(-1) = -0.25$

$f(1) = -0.16$

Test point $x = 4$

$$f(4) = \frac{4^2}{4^2 - 4 - 6}$$

$$= \frac{16}{6}$$

$$= 2.7$$

Slant Asymptotes

Given $f(x) = \dfrac{N(x)}{D(x)}$

If the degree of $N(x)$ is bigger than the degree of $D(x)$ by 1 then the graph of $f(x)$ has a "slant asymptote" and it can be found via Division of polynomials.

Example: Find the slant Asymptote of $f(x) = \dfrac{x^2 - x + 3}{x+1}$

Solution:

$$f(x) = \frac{x^{2} - x + 3}{x^{1} + 1}$$

Degree 2

Degree 1

$2 > 1$ by 1

Then $f(x)$ has a slant asymptote

Divide $(x^2 - x + 3) \div (x + 1)$

Opposite

$$\begin{array}{r|rrr} & 1 & -1 & 3 \\ & & -1 & 2 \\ \hline -1 & 1 & -2 & \boxed{5} \end{array}$$

$$x - 2 + \frac{5}{x+1}$$

Slant Asymptote

Slant Asymptote

$$y = x - 2$$